Madras College Maths Department
Higher Maths
Apps 1.4 Applying Differential Calculus

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Written solutions for each exercise are available at

http://madrasmaths.com/courses/higher/revison_materials_higher.html

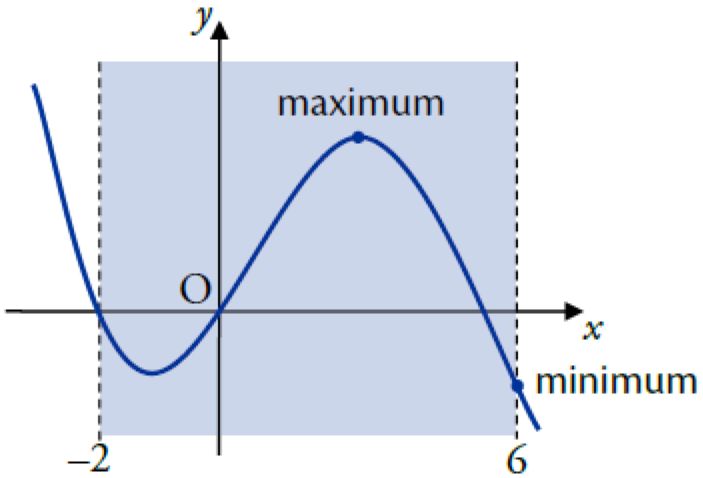
You should check your solutions at the end of each exercise and ask your teacher or attend study support if there any problems.

Closed Intervals

Sometimes it is necessary to restrict the part of the graph we are looking at using a **closed interval** (also called a restricted domain).

The maximum and minimum y-values can either be at stationary points or at the end points of the closed interval.

Below is a sketch of a curve with the closed interval $-2 \leq x \leq 6$ shaded.



Notice that the minimum value occurs at one of the end points in this example. It is important to check for this.

EXAMPLE

A function f is defined for $-1 \leq x \leq 4$ by $f(x) = 2x^3 - 5x^2 - 4x + 1$.
Find the maximum and minimum value of $f(x)$.

Optimisation

In the section on closed intervals, we saw that it is possible to find maximum and minimum values of a function.

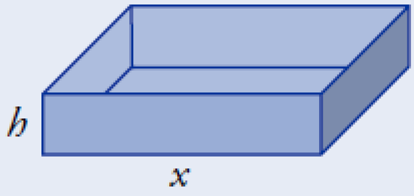
This is often useful in applications; for example a company may have a function $P(x)$ which predicts the profit if £x is spent on raw materials – the management would be very interested in finding the value of x which gave the maximum value of $P(x)$.

The process of finding these optimal values is called **optimisation**.

Sometimes you will have to find the appropriate function before you can start optimisation.

EXAMPLE

1. Small wooden trays, with open tops and square bases, are being designed. They must have a volume of 108 cubic centimetres.



The internal length of one side of the base is x centimetres, and the internal height of the tray is h centimetres.

(a) Show that the total internal surface area A of one tray is given by

$$A = x^2 + \frac{432}{x}.$$

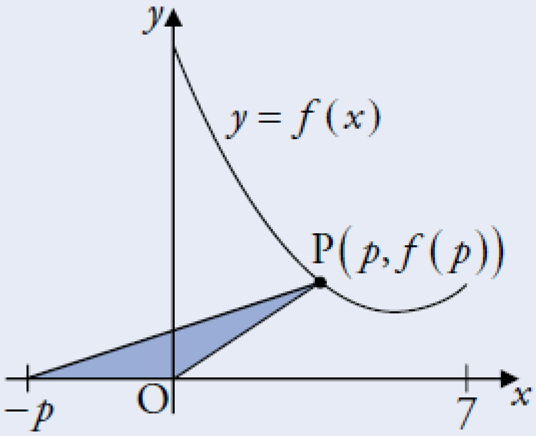
(b) Find the dimensions of the tray using the least amount of wood.

Optimisation with closed intervals

In practical situations, there may be bounds on the values we can use. For example, the company from before might only have £100 000 available to spend on raw materials. We would need to take this into account when optimising.

Recall from the section on Closed Intervals that the maximum and minimum values of a function can occur at turning points *or* the endpoints of a closed interval.

2. The point P lies on the graph of $f(x) = x^2 - 12x + 45$, between $x = 0$ and $x = 7$.



A triangle is formed with vertices at the origin, P and $(-p, 0)$.

(a) Show that the area, A square units, of this triangle is given by

$$A = \tfrac{1}{2}p^3 - 6p^2 + \tfrac{45}{2}p.$$

(b) Find the greatest possible value of A and the corresponding value of p for which it occurs.

Rates of Change

The derivative of a function describes its "rate of change". This can be evaluated for specific values by substituting them into the derivative.

EXAMPLES

1. Given $f(x) = 2x^5$, find the rate of change of f when $x = 3$.

2. Given $y = \dfrac{1}{x^{\frac{2}{3}}}$ for $x \neq 0$, calculate the rate of change of y when $x = 8$.

Displacement, velocity and acceleration

The velocity v of an object is defined as the rate of change of displacement s with respect to time t. That is:

$$v = \frac{ds}{dt}.$$

Also, acceleration a is defined as the rate of change of velocity with respect to time:

$$a = \frac{dv}{dt}.$$

EXAMPLE

3. A ball is thrown so that its displacement s after t seconds is given by
$s(t) = 12t - 5t^2$.
Find its velocity after 2 seconds.

Geometric Interpretation of Integration

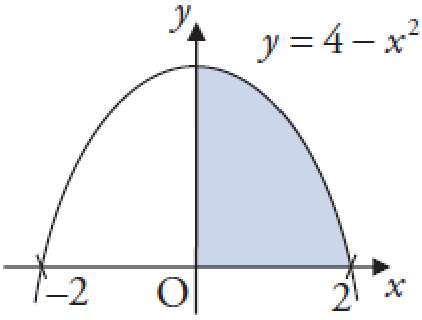
We will now consider the meaning of integration in the context of areas.

Consider $\int_0^2 (4 - x^2)\, dx = \left[4x - \frac{1}{3}x^3\right]_0^2$

$$= \left(8 - \tfrac{8}{3}\right) - 0$$

$$= 5\tfrac{1}{3}.$$

On the graph of $y = 4 - x^2$:



The shaded area is given by $\int_0^2 (4 - x^2)\, dx$.

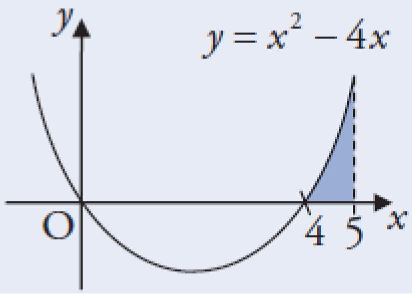
Therefore the shaded area is $5\frac{1}{3}$ square units.

In general, the area enclosed by the graph $y = f(x)$ and the x-axis, between $x = a$ and $x = b$, is given by

$$\int_a^b f(x)\, dx.$$

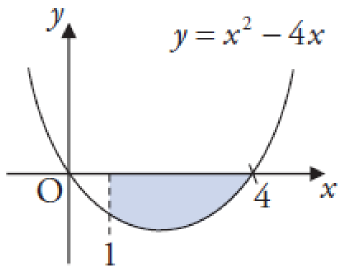
EXAMPLE

1. The graph of $y = x^2 - 4x$ is shown below. Calculate the shaded area.



Areas below the x-axis

Care needs to be taken if part or all of the area lies below the x-axis. For example if we look at the graph of $y = x^2 - 4$:



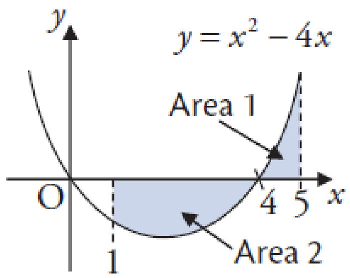
The shaded area is given by

$$\int_1^4 (x^2 - 4x)\,dx = \left[\frac{x^3}{3} - \frac{4x^2}{2}\right]_1^4$$
$$= \left(\frac{4^3}{3} - 2(4)^2\right) - \left(\tfrac{1}{3} - 2\right)$$
$$= \tfrac{64}{3} - 32 - \tfrac{1}{3} + 2$$
$$= \tfrac{63}{3} - 30 = 21 - 30 = -9.$$

In this case, the negative indicates that the area is below the x-axis, as can be seen from the diagram. The area is therefore 9 square units.

Areas above and below the x-axis

Consider the graph from the example above, with a different shaded area:



From the examples above, the total shaded area is:

$$\text{Area 1} + \text{Area 2} = 2\tfrac{1}{3} + 9 = 11\tfrac{1}{3} \text{ square units.}$$

Using the method from above, we might try to calculate the shaded area as follows:

$$\int_1^5 (x^2 - 4x)\,dx = \left[\frac{x^3}{3} - \frac{4x^2}{2}\right]_1^5$$
$$= \left(\frac{5^3}{3} - 2(5)^2\right) - \left(\tfrac{1}{3} - 2\right)$$
$$= \tfrac{125}{3} - 50 - \tfrac{1}{3} + 2$$
$$= \tfrac{124}{3} - 48 = -6\tfrac{2}{3}.$$

Clearly this shaded area is not $6\frac{2}{3}$ square units since we already found it to be $11\frac{1}{3}$ square units. This problem arises because Area 1 is above the x-axis, while Area 2 is below.

To find the true area, we needed to evaluate two integrals:

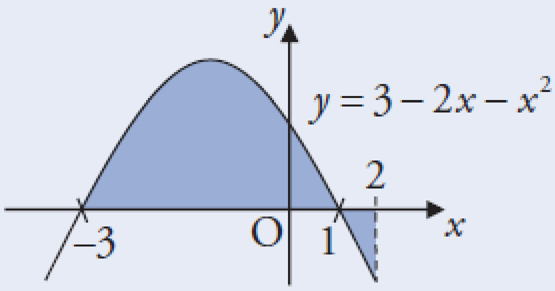
$$\int_1^4 (x^2 - 4x)\,dx \qquad \text{and} \qquad \int_4^5 (x^2 - 4x)\,dx.$$

We then found the total shaded area by adding the two areas together.

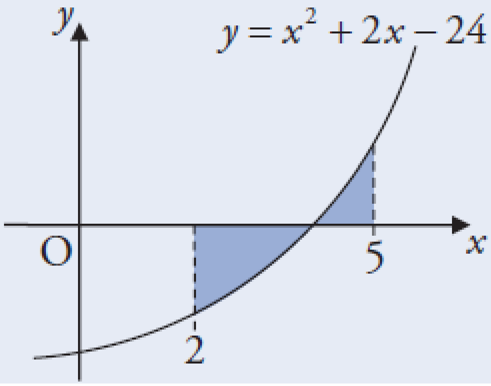
We must take care to do this whenever the area is split up in this way.

EXAMPLES

2. Calculate the shaded area shown in the diagram below.



3. Calculate the shaded area shown in the diagram below.

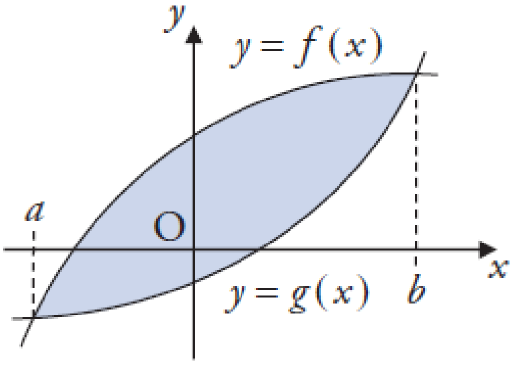


Areas between Curves

The area between two curves between $x = a$ and $x = b$ is calculated as:

$$\int_a^b (\text{upper curve} - \text{lower curve})\, dx \text{ square units.}$$

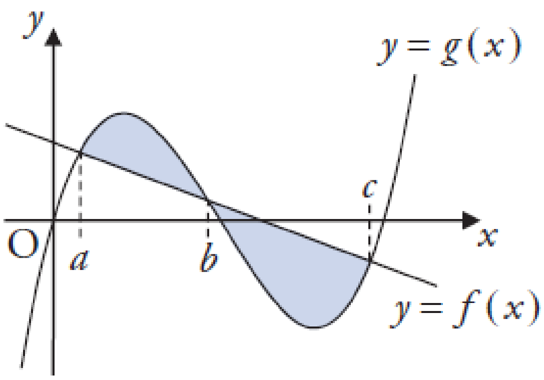
So for the shaded area shown below:



The area is $\int_a^b (f(x) - g(x))\, dx$ square units

When dealing with areas between curves, areas above and below the x-axis do not need to be calculated separately.

However, care must be taken with more complicated curves, as these may give rise to more than one closed area. These areas must be evaluated separately. For example:



In this case we apply $\int_a^b (\text{upper curve} - \text{lower curve})\, dx$ to each area.

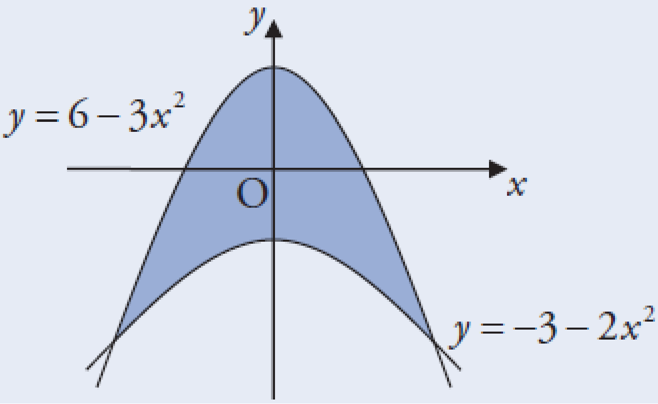
So the shaded area is given by:

$$\int_a^b (g(x) - f(x))\, dx + \int_b^c (f(x) - g(x))\, dx.$$

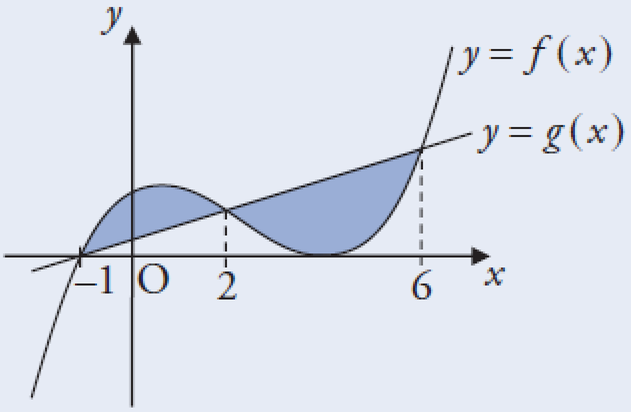
Additional examples are available in the Dynamic Maths Study Notes available at http://madrasmaths.com/courses/higher/revison_materials_higher.html (password: madrasmaths) and at hsn.uk.net

EXAMPLES

1. Calculate the shaded area enclosed by the curves with equations $y = 6 - 3x^2$ and $y = -3 - 2x^2$.

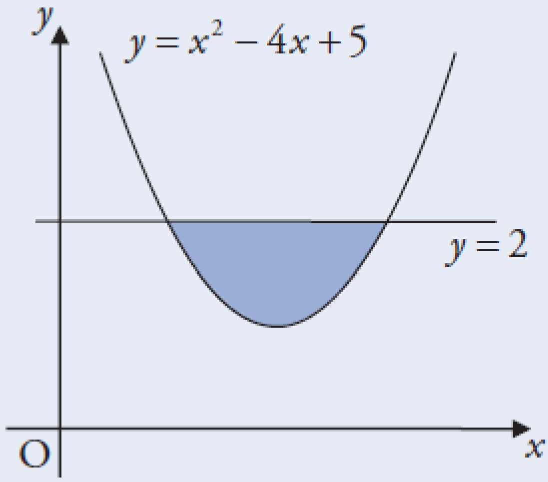


2. Two functions are defined for $x \in \mathbb{R}$ by $f(x) = x^3 - 7x^2 + 8x + 16$ and $g(x) = 4x + 4$. The graphs of $y = f(x)$ and $y = g(x)$ are shown below.



Calculate the shaded area.

3. A trough is 2 metres long. A cross-section of the trough is shown below.



The cross-section is part of the parabola with equation $y = x^2 - 4x + 5$.

Find the volume of the trough.

Practice Unit Assessments

Practice test 1

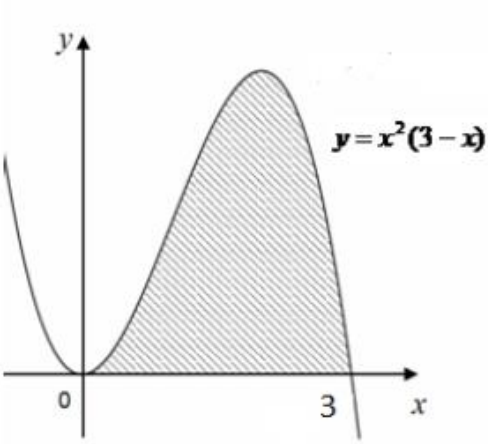
1 A box with a square base and open top has a surface area of 768 cm^2. The volume of the box can be represented by the formula:

$$V(x) = 75x - \frac{1}{9}x^3$$

Find the value of x which maximises the volume of the box.

(5)

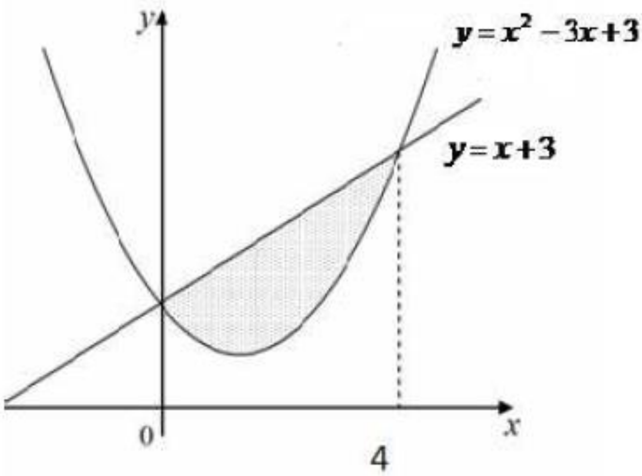
2 The curve with equation $y = x^2(3-x)$ is shown in the diagram.



Calculate the shaded area.

(#2.1 + 4)

3 The line with equation $y = x + 3$ meets the curve with equation $y = x^2 - 3x + 3$ when $x = 0$ and $x = 4$ as shown in the diagram.



Calculate the shaded area.

(5)

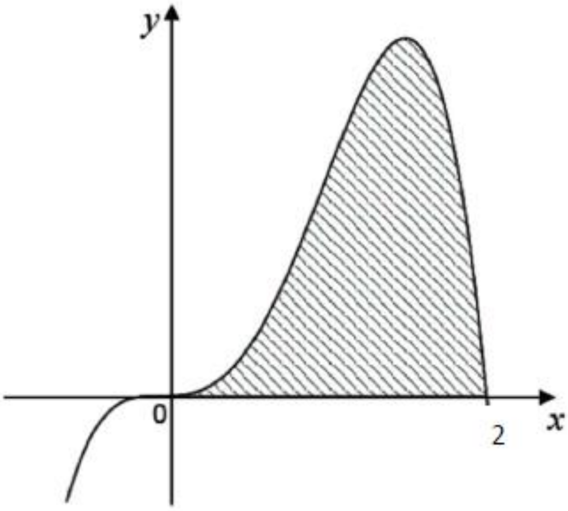
Practice test 2

1 The area of a rectangle can be represented by the formula
$$A(x) = 27x - 3x^3, \text{ where } x > 0.$$
Find the value of x which maximises the area of the rectangle.
Justify your answer.

(5)

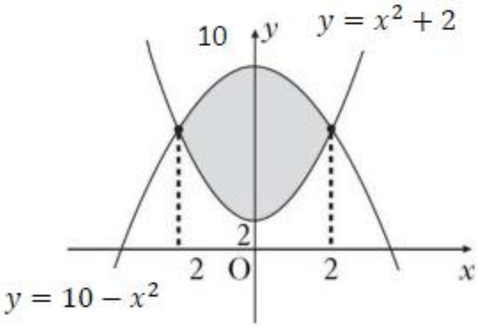
2 The curve with equation $y = x^3(2 - x)$ is shown in the diagram.

Calculate the shaded area.



(4)

3 The diagram shows graphs with equations $y = 10 - x^2$ and $y = x^2 + 2$



(a) Which of the following integrals represents the shaded area?

A $\int_2^{10}(2x^2 - 8)\,dx$ B $\int_{-2}^{2}(8 - 2x^2)\,dx$ C $\int_{-2}^{2}(2x^2 - 8)\,dx$ D $\int_2^{10}(8 - 2x^2)\,dx$

(1)

(b) Calculate the shaded area.

(3)

Practice Unit Assessment Solutions

① $V'(x) = 75 - \frac{3}{9}x^2$

$V'(x) = 0$ for stationary point

$75 - \frac{3}{9}x^2 = 0$

$75 = \frac{1}{3}x^2$

$225 = x^2$

$x = \pm 15 \quad x > 0$

so $\underline{x = 15}$

x	$\rightarrow$	15	$\rightarrow$
$V'(x) = 75 - \frac{1}{3}x^2$	+	0	−
slope	/	—	\

max value of $x = 15$

② $\int_0^3 x^2(3-x)\,dx$

$= \int_0^3 3x^2 - x^3\,dx$

$= \left[x^3 - \frac{x^4}{4}\right]_0^3$

$= \left((3)^3 - \frac{(3)^4}{4}\right) - \left((0)^3 - \frac{(0)^4}{4}\right)$

$= \underline{\frac{27}{4} \text{ square units.}}$

③ Area $= \int_0^4 (\text{top curve} - \text{bottom curve})\,dx$

$\cancel{= \int_0^4 (x^2 - 3x + 3) - (x+3)\,dx}$

$\cancel{= \int_0^4 x^2 - 4x\,dx}$

$\cancel{= \left[\frac{x^3}{3} - 2x^2\right]_0^4}$

$\cancel{= \left(\frac{(4)^3}{3} - 2(4)^2\right) - \left(\frac{0^3}{3} - 2(0)^2\right)}$

$= \int_0^4 (x+3) - (x^2 - 3x + 3)\,dx$

$= \int_0^4 -x^2 + 4x\,dx$

$= \left[-\frac{x^3}{3} + 2x^2\right]_0^4$

$= \left(-\frac{(4)^3}{3} + 2(4)^2\right) - \left(-\frac{(0)^3}{3} + 2(0)^2\right)$

$= \left(-\frac{64}{3} + 32\right) - (0)$

$= \underline{\frac{32}{3} \text{ square units}}$

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Test 2

① $A'(x) = 27 - 9x^2$

$A'(x) = 0$ for stationary point

$27 - 9x^2 = 0$

$27 = 9x^2$

$x^2 = 3$

$x = \pm\sqrt{3}, \ x > 0$

$x = \sqrt{3}$

x	$\rightarrow$	$\sqrt{3}$	$\rightarrow$
$A'(x) = 27 - 9x^2$	+	0	−
slope	/	—	\

$x = \sqrt{3}$ maximises the area of the rectangle.

② $\int_0^2 x^3(2-x)\,dx$

$= \int_0^2 2x^3 - x^4\,dx$

$= \left[\frac{x^4}{2} - \frac{x^5}{5}\right]_0^2$

$= \left[\frac{(2)^4}{2} - \frac{(2)^5}{5}\right] - \left(\cancel{\frac{(0)^4}{2} - \frac{(0)^5}{5}}\right)$ 0

$= \frac{8}{5}$ square units

③(a) $\int_{-2}^{2}$ top curve − bottom curve dx

$= \int_{-2}^{2} (10 - x^2) - (x^2 + 2)\,dx$

$= \int_{-2}^{2} 8 - 2x^2\,dx$

(b) $\left[8x - \frac{2x^3}{3}\right]_{-2}^{2}$

$= \left(8(2) - \frac{2(2)^3}{3}\right) - \left(8(-2) - \frac{2(-2)^3}{3}\right)$

$= \frac{64}{3}$ square units